# ON TRAVELLING WAVES OF GAS DYNAMIC EQUATIONS

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1. In a previous paper [1] consideration was given to plane travelling wave systems associated with quasi-linear equations such as

$$a_{ijk}(u_1, \ldots, u_m)\frac{\partial u_j}{\partial x_k} = 0 \qquad (i, j, k = 1, \ldots, m) \tag{1.1}$$

We call the travelling wave of rank $r$ a solution of system (1.1) if for it the $m - r$ functional dependencies are satisfied:

$$\varphi_\alpha(u_1, \ldots, u_m) = 0 \qquad (\alpha = 1, \ldots, m - r) \tag{1.2}$$

In the given classification the travelling wave of rank 1 coincides with the plane travelling wave [1].

In the present work are considered travelling waves of rank $m - 1$. The following algorism (formal treatment) is suggested to find them.

Let

$$u_m = \varphi(u_1, \ldots, u_{m-1}) \tag{1.3}$$

be the functional dependence determining the travelling wave of rank $m - 1$.

From equation (1.3) there follows that $u_\alpha(x_1, \ldots, x_m)$, $\alpha = 1, \ldots, m$ functions have common level lines. Let these lines satisfy the differential equation

$$\frac{dx_i}{\Delta_i} = \frac{dx_m}{1} \qquad (i = 1, \ldots, m - 1) \tag{1.4}$$

where $\Delta_i$ represents any of several functions of $x_1, \ldots, x_m$. For any function $f(u_1, \ldots, u_m)$ we must have

$$\frac{\partial f}{\partial x_m} + \Delta_k \frac{\partial f}{\partial x_k} = 0 \qquad (k = 1, \ldots, m - 1) \tag{1.5}$$

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The relations (1.3), (1.5) allow us to eliminate in system (1.1) the function $u_m$ and the derivatives of $x_m$.

Substituting (1.3), (1.5) into (1.1), we obtain the system in which

$$L_i \equiv A_{i\alpha\beta} \frac{\partial u_\alpha}{\partial x_\beta} = 0 \qquad \begin{pmatrix} i = 1, \dots, m \\ \alpha, \beta = 1, \dots, m-1 \end{pmatrix} \tag{1.6}$$

where

$$A_{i\alpha\beta} = a_{i\alpha\beta} - a_{i\alpha m}\Delta_\beta + a_{im\beta}\varphi_\alpha - a_{imm}\varphi_\alpha\Delta_\beta, \quad \varphi_\alpha = \frac{\partial\varphi}{\partial u_\alpha} \tag{1.7}$$

System (1.6) represents an overdetermined system into whose coefficients enters as a parameter the variable $x_m$. When expressing the condition that equation (1.6) apply to any $x_m$ it is necessary to add to (1.6) the following equation:

$$\partial L_i / \partial x_m = 0 \tag{1.8}$$

Noting that relations

$$\Delta_\gamma \frac{\partial L_i}{\partial x_\gamma} = 0 \qquad (\gamma = 1, \dots, m-1) \tag{1.9}$$

follow from (1.6) it is seen that equation (1.8) may be written in the form

$$\delta L_i = 0 \tag{1.10}$$

$$\delta = \frac{\partial}{\partial x_m} + \Delta_\gamma \frac{\partial}{\partial x_\gamma} \qquad (\gamma = 1, \dots, m-1) \tag{1.11}$$

Taking into account (1.5) we obtain the relation

$$\delta \frac{\partial u_\alpha}{\partial x_\beta} = -\frac{\partial \Delta_\gamma}{\partial x_\beta} \frac{\partial u_\alpha}{\partial x_\gamma} \tag{1.12}$$

Accordingly, equation (1.10) takes the following form

$$A_{i\alpha\beta}{}^{(1)} \frac{\partial u_\alpha}{\partial x_\beta} \equiv \left(\delta A_{i\alpha\beta} - A_{i\alpha\gamma} \frac{\partial \Delta_\beta}{\partial x_\gamma}\right) \frac{\partial u_\alpha}{\partial x_\beta} = 0 \tag{1.13}$$

Further consequences

$$\frac{\partial^s L_i}{\partial x_m{}^s} = 0 \qquad (s = 2, \dots) \tag{1.14}$$

lead to the equation

$$A_{i\alpha\beta}{}^{(s)} \frac{\partial u_\alpha}{\partial x_\beta} = 0 \tag{1.15}$$

where

$$A_{i\alpha\beta}{}^{(s)} = \delta A_{i\alpha\beta}{}^{(s-1)} - A_{i\alpha\gamma}{}^{(s-1)} \frac{\partial \Delta_\beta}{\partial x_\gamma} \tag{1.16}$$

If $\Delta_\gamma$ is regarded as a direct function of $x_1, \dots, x_m$, then conditions (1.15) represent quasi-linear equations for $u_\alpha$. If $\Delta_\gamma$ appear as functions of

$$u_1, \dots, u_{m-1}, \quad x_1, \dots, x_m,$$

then equation (1.5) will be of power $s + 1$ with regard to derivatives

$\partial u_\alpha/\partial x_\beta$. Particularly when $\Lambda_\gamma$ are functions $u_1, \ldots, u_{m-1}$, then expressions $A_{i\alpha\beta}^{(s)}\partial u_\alpha/\partial x_\beta$ will be in the form of powers $s+1$ with regard to the derivatives $\partial u_\alpha/\partial x_\beta$.

A further problem appears to be the investigation of the combatibility of system (1.15) where one can formally assume

$$A_{i\alpha\beta}^{(0)} = A_{i\alpha\beta}$$

In this manner, system (1.15) will include equations (1.6) and all consequences from them and determine the travelling wave.

Depending on the degree of arbitrariness which we will require from the solution

$$u_i = u_i(x_1, \ldots, x_m),$$

we will obtain various specific limitations on $\phi_\alpha$, $\Lambda_\beta$ and thereby various systems (1.15).

The arbitrariness of the solution will materially depend on the rank of the systems (1.15). By this we mean the number of really independent equations in system (1.15).

An analogous treatment (algorism) may be suggested in the case of travelling waves of arbitrary rank.

2. We consider the gas dynamics equation of a polytropic gas:

$$\rho\left(\frac{\partial u_i}{\partial t} + u_k\frac{\partial u_i}{\partial x_k}\right) + \frac{\partial p}{\partial x_i} = 0 \qquad \left(p = \frac{a^2\rho^\gamma}{\gamma}\right) \tag{2.1}$$

$$\frac{\partial \rho}{\partial t} + u_k\frac{\partial \rho}{\partial x_k} + \rho\frac{\partial u_k}{\partial x_k} = 0 \tag{2.2}$$

In the adiabatic case, when $a^2 = a^2(s) \equiv \text{const}$, considering the variables $u_i$, $\theta = [a^2/(\gamma-1)]\rho^{\gamma-1}$, we obtain

$$\frac{\partial u_i}{\partial t} + u_k\frac{\partial u_i}{\partial x_k} + \frac{\partial \theta}{\partial x_i} = 0, \qquad \frac{\partial \theta}{\partial t} + u_k\frac{\partial \theta}{\partial x_k} + (\gamma-1)\,\theta\frac{\partial u_k}{\partial x_k} = 0 \tag{2.3}$$

In the isothermal case $p = a^2\rho$, $a^2 = RT = \text{const}$, $\gamma = 1$, $\theta = \ln\rho$ and equation (2.3) assumes the form

$$\frac{\partial u_i}{\partial t} + u_k\frac{\partial u_i}{\partial x_k} + a^2\frac{\partial \theta}{\partial x_i} = 0, \quad \frac{\partial \theta}{\partial t} + u_k\frac{\partial \theta}{\partial x_k} + \frac{\partial u_k}{\partial x_k} = 0 \tag{2.4}$$

The systems (2.3), (2.4) belong to type (1.1), therefore to them may be applied the formal treatment (algorism) of the determination of travelling waves.

We consider a travelling wave of rank 2 of plane motion ($m = 3$) of a polytropic gas.

In our case $\theta$ plays the role of variable $u_3$, and $t$ that of variable $x_3$; system (1.6) takes the form

$$(u_1-\Delta_1+\varphi_1)\frac{\partial u_1}{\partial x_1}+(u_2-\Delta_2)\frac{\partial u_1}{\partial x_2}+\varphi_2\frac{\partial u_2}{\partial x_1}+0\frac{\partial u_2}{\partial x_2}=0$$

$$0\frac{\partial u_1}{\partial x_1}+\varphi_1\frac{\partial u_1}{\partial x_2}+(u_1-\Delta_1)\frac{\partial u_2}{\partial x_1}+(u_2-\Delta_2+\varphi_2)\frac{\partial u_2}{\partial x_2}=0 \qquad (2.5)$$

$$[(\gamma-1)\varphi+\varphi_1(u_1-\Delta_1)]\frac{\partial u_1}{\partial x_1}+\varphi_1(u_2-\Delta_2)\frac{\partial u_1}{\partial x_2}+\varphi_2(u_1-\Delta_1)\frac{\partial u_2}{\partial u_1}+$$
$$+[(\gamma-1)\varphi+\varphi_2(u_2-\Delta_2)]\frac{\partial u_2}{\partial x_2}=0$$

Let us require that with the function $\phi\ (u_1,\ u_2)$ fixed the travelling wave possesses arbitrariness of two functions of one argument. For this it is necessary that the rank of the system (2.5) be equal to 2. From this condition we arrive at two possibilities:

(a) $$\varphi_\alpha=\Delta_\alpha-u_\alpha=a_\alpha\ (\alpha=1,2) \qquad (2.6)$$

(b) $$a_1\varphi_1+a_2\varphi_2=0$$

$$(a_1^2+a_2^2)[\varphi_1^2+\varphi_2^2-(\gamma-1)\varphi]-(\gamma-1)\varphi(\varphi_1^2+\varphi_2^2)=0 \qquad (2.7)$$

We limit ourselves to case (a) which appears basic. From (2.6) it follows that the lines of level are straight and

$$\Delta_\alpha=\frac{\partial\Delta}{\partial u_\alpha},\qquad \Delta=\varphi+\frac{u_1^2+u_2^2}{2} \qquad (2.8)$$

System (2.5) is reduced to two equations:

$$L_i=A_{i\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta}=0 \qquad (i,\ \alpha,\ \beta=1,\ 2) \qquad (2.9)$$

where $$A_{111}=A_{122}=0,\qquad A_{112}=-A_{121}=1 \qquad (2.10)$$
$$A_{2\alpha\beta}=(\gamma-1)\varphi\delta_{\alpha\beta}-\varphi_\alpha\varphi_\beta \qquad (\alpha,\ \beta=1,\ 2)$$

($\delta_{\alpha\beta}$ — is the Kronecker symbol)

Equation $L_1 = 0$ means that the motion is a potential motion. Conditions $\delta L_1 = 0$ yield

$$A_{i\alpha\beta}\Delta_{\gamma\sigma}\frac{\partial u_\alpha}{\partial x_\gamma}\frac{\partial u_\sigma}{\partial x_\beta}=0,\qquad \Delta_{\gamma\sigma}=\frac{\partial^2\Delta}{\partial u_\gamma\partial u_\sigma} \qquad (2.11)$$

Or on writing $\partial\psi/\partial x_a = u_a$ where $\psi$ is the potential function, we have

$$A_{i\alpha\beta}\Delta_{\gamma\sigma}\psi_{\alpha\gamma}\psi_{\sigma\beta}=0 \qquad (2.12)$$

Conditions $\delta^2 L_i = 0$ have the form

$$A_{i\alpha\beta}\Delta_{\gamma\sigma}\Delta_{\rho\omega}\psi_{\alpha\gamma}\psi_{\beta\rho}\psi_{\sigma\omega}=0 \qquad (2.13)$$

It is easy to see that all conditions

$$\delta^s L_1=0 \qquad (2.14)$$

are fulfilled identically.

Let us require that equation $\delta L_2 = 0$ follows from (2.9). For this it is necessary and sufficient that the quadratic form $\delta L_2$ be divisible into a linear form $L_2$. Using equation (2.9), condition $\delta L_2 = 0$ may be reduced to the form

$$KL(\varphi) = 0 \tag{2.15}$$

where

$$K = \psi_{11}\psi_{22} - \psi_{12}^2 \tag{2.16}$$

$$L(\varphi) = (\varphi_{11} + 1)[(\gamma - 1)\varphi - \varphi_2^2] + 2\varphi_1\varphi_2\varphi_{12} + [(\gamma - 1)\varphi - \varphi_1^2](\varphi_{22} + 1) = 0 \tag{2.17}$$

The case $K = 0$, as is easy to see, leads to the plane travelling wave of rank 1. In this manner, we obtain for $\phi$ a quasi-linear equation of the second order:

$$L(\varphi) = 0 \tag{2.18}$$

Condition $\delta^2 L_2 = 0$ with consideration of equations $L_2 = 0$, $\delta L_2 = 0$ takes the form

$$KK_1L_2 = 0 \qquad (K_1 = \Delta_{11}\Delta_{22} - \Delta_{12}^2) \tag{2.19}$$

It follows at once from this that conditions $\delta^s L_2 = 0$, $s > 1$ do not give anything new and equation (2.18) seems a sufficient condition in order that solution $u_i(x_1, x_2, t)$ possesses arbitrariness of two functions of one argument.

The motion corresponding to the given solution $\phi$ $(u_1, u_2)$ of equation (2.18) may be obtained in the following manner. Let

$$U_1 = U_1(x_1, x_2) \quad U_2 = U_2(x_1, x_2)$$

be the solution of system (2.9), $\Delta(U_1, U_2)$ the function corresponding to $\phi$ $(U_1, U_2)$.

Let us draw through every point $x_{10}$, $x_{20}$ of the surface $t = t_0$ a ray

$$\frac{x_1 - x_{10}}{\Delta_1[U_1(x_{10}, x_{20}), U_2(x_{10}, x_{20})]} = \frac{x_2 - x_{20}}{\Delta_2[U_1(x_{10}, x_{20}), U_2(x_{10}, x_{20})]} = \frac{t - t_0}{1} \tag{2.20}$$

Along each ray, going through the point $x_{10}$, $x_{20}$, $t_0$ we will assume

$$u_\alpha(x_1, x_2, t) = U_\alpha(x_{10}, x_{20}) \tag{2.21}$$

In addition to this, everywhere $\theta = \phi$ $(U_1, U_2)$. Then the functions

$$u_\alpha(x_1, x_2, t), \qquad \theta(x_1, x_2, t) = \varphi[u_1(x_1, x_2, t), \quad u_2(x_1, x_2, t)]$$

determine the desired travelling wave. As a consequence of the assumption $K \neq 0$ for equation (2.9), one can apply the holographic transformation to the equations (2.9)

$$\frac{\partial x_\alpha}{\partial u_\beta} = (-1)^{\alpha+\beta} \frac{1}{K} \frac{\partial u_{3-\alpha}}{\partial x_{3-\beta}} \tag{2.22}$$

Since the motion is potential, then

$$\partial x_1 / \partial u_2 = \partial x_2 / \partial u_1$$

and one can introduce the potential function $X(u_1, u_2)$ such that

$$\frac{\partial X}{\partial u_\alpha} = x_\alpha(u_1, u_2) \tag{2.23}$$

It is clear that $X(u_1, u_2)$ satisfies equation

$$[(\gamma-1)\varphi - \varphi_1^2] \frac{\partial^2 X}{\partial u_2^2} + 2\varphi_1\varphi_2 \frac{\partial^2 X}{\partial u_1 \partial u_2} + [(\gamma-1)\varphi - \varphi_2^2] \frac{\partial^2 X}{\partial u_1^2} = 0 \tag{2.24}$$

In this manner, the following theorem is true:

*Theorem 1.* If $\phi(u_1, u_1)$ satisfies the equation (2.18), then the corresponding travelling wave possesses arbitrariness of two functions of one argument and is expressed by means of formulae (2.20) to (2.23) through an integral of $X(u_1, u_2)$ of equation (2.24).

3. Let us call conical flow the travelling wave of rank 2 in which all the straight level lines pass through a single point $x_{10}$, $x_{20}$, $t_0$, of the phase space $x_1$, $x_2$, $t$. In other words, the congruence of straight level lines turns out to be conical. Let us prove that the resulting travelling waves do not appear, generally speaking, as conical.

The conical congruence has the following infinitesimal characteristic: any straight congruence is intersected by any straight line which goes through its infinitesimal neighborhood. Expressing this fact we obtain the following conditions of conical flow:

$$\frac{\partial \Delta_1}{\partial x_2} = \frac{\partial \Delta_2}{\partial x_1} = 0, \qquad \frac{\partial \Delta_1}{\partial x_1} = \frac{\partial \Delta_2}{\partial x_2} \tag{3.1}$$

From this follow the equations:

$$\begin{aligned} \Delta_{11} \frac{\partial u_1}{\partial x_2} + \Delta_{12} \frac{\partial u_2}{\partial x_2} &= 0 \\ \Delta_{12} \frac{\partial u_1}{\partial x_1} + \Delta_{22} \frac{\partial u_2}{\partial x_1} &= 0 \\ \Delta_{11} \frac{\partial u_1}{\partial x_1} - \Delta_{22} \frac{\partial u_2}{\partial x_2} &= 0 \end{aligned} \tag{3.2}$$

In order that the conditions for being conical be satisfied for any travelling waves corresponding to the given function $\phi(u_1, u_2)$, it is necessary and sufficient that equations (3.2) follow from equation (2.9) i.e., that the rank of matrix $\|M\|$ be equal to 2, where

$$\|M\| = \left\| \begin{matrix} 0 & 1 & -1 & 0 \\ (\gamma-1)\varphi - \varphi_1^2 & -\varphi_1\varphi_2 & -\varphi_1\varphi_2 & (\gamma-1)\varphi - \varphi_2^2 \\ \Delta_{12} & 0 & \Delta_{22} & 0 \\ 0 & \Delta_{11} & 0 & \Delta_{12} \\ \Delta_{11} & 0 & 0 & -\Delta_{22} \end{matrix} \right\|$$

This is only possible for $\Delta_{\alpha\beta} = 0$, $\alpha, \beta = 1, 2$. From this, taking into account (2.6), we obtain

$$\varphi_{11} + 1 = 0, \qquad \varphi_{12} = 0, \qquad \varphi_{22} + 1 = 0 \tag{3.3}$$

$$\varphi = c_0 + c_1 u_1 + c_2 u_2 - \tfrac{1}{2}(u_1^2 + u_2^2), \qquad \Delta = c_0 + c_1 u_1 + c_2 u_2 \tag{3.4}$$

In the general case $\phi$ does not satisfy conditions (3.3) and the conditions for being conical (3.2) represent substantially new equations. In order that conical flow be not trivial it is necessary to fulfil the condition; rank of $\|M\| = 3$.

It is easy to see that all the values of the fourth order matrix $M$ will be equal to zero by virtue of conditions (2.18).

In this manner, for any solution $\phi$ of equation $L(\phi) = 0$, except (3.4), the matrix of coefficients of equation (2.9), (3.2) has the rank 3. From this we have

$$\frac{\partial u_\alpha}{\partial x_\beta} = (-1)^{\alpha+\beta}\Delta_{3-\alpha,3-\beta}\ \mu \qquad (\alpha, \beta = 1, 2) \tag{3.5}$$

where $\mu$ is a certain multiplying factor.

Applying the hodographic transformation to (3.5), we obtain

$$\frac{\partial x_\alpha}{\partial u_\beta} = \mu_1 \frac{\partial \Delta_\alpha}{\partial u_\beta} \qquad (\alpha, \beta = 1, 2) \tag{3.6}$$

From this follows at once $\mu$ = const = $c$, and

$$x_\alpha = c\Delta_\alpha + c_\alpha \qquad (\alpha = 1, 2), \qquad X = c\Delta + c_1 u_1 + c_2 u_2 + c_3 \tag{3.7}$$

It is easy to see that equations (2.18), (2.24) are satisfied. In this manner, the following theorem is proved.

*Theorem* 2. In the case when

$$\varphi = c_0 + c_1 u_1 + c_2 u_2 - {}^1/_2 (u_1^2 + u_2^2),$$

all travelling waves are conical flows. For the remaining solutions $\phi$ of $L(\phi) = 0$ the motions, generally speaking, do not turn out to be conical, but for any $\phi$ in the class of corresponding travelling waves there exists a completely determinate conical flow.

4. Let us apply the results obtained to the solution of the problem of gas motion bounded by two surfaces. Let the space $x_1$, $x_2$, $x_3$ be an infinite volume of stagnant gas, enclosed at the instant where $t < 0$ inside the corner between planes $x_1 = 0$, $x_2 = 0$. At the instant $t = 0$ the planes begin to move according to the law:

$$x_i = f_i(t) \qquad (i = 1, 2) \tag{4.1}$$

It is clear that the motion will be two dimensional, not depending on the coordinate $x_3$. In the following we will identify plane $x_3 = 0$ with the plane of the diagram, planes $x_i$ = const. will accordingly be represented by the coordinate lines.

Upon first examination we will assume function $f_i(t)$ to be such that until the instant of time $T$ there will be no strong discontinuities in the motion. Then, at a certain instant of time $t < T$ in the plane $x_1$, $x_2$ we will have the following picture of the motion (Fig.1).

In region I we have stagnant gas

$$u_1 = u_2 = 0, \qquad \theta = \theta_0 = c_0^2/(\gamma-1) \tag{4.2}$$

In region II (vertical strip above AC) we have one-dimensional motion, not depending on $x_2$ and proving to be a plane travelling wave (wave of rank 1), i.e., a Riemann wave, to which the well-known relationships apply:

$$u_2 = 0, \qquad u_1 = g_1[x_1 - (u_1 + c)t], \qquad u_1 - \frac{2}{\gamma-1}c = -\frac{2}{\gamma-1}c_0 \tag{4.3}$$

Line $\gamma_1$, dividing region II from region I will be straight $x_1 = c_0 t$. In region III (horizontal strip to the right of BC) we also have one dimensional motion, not depending on $x_1$. This also is a Riemann wave:

$$u_1 = 0, \qquad u_2 = g_2[x_2 - (u_2 + c)t], \qquad u_2 - \frac{2}{\gamma-1}c = -\frac{2}{\gamma-1}c_0 \tag{4.4}$$

The line $\gamma_2$, the boundary between regions I and III, is straight. $x_2 = c_0 t$. In region IV we will look for motion of the type of travelling wave of rank 2.

Since for any function $\phi(u_1, u_2)$ that is the solution of equation $L(\phi) = 0$, the travelling wave of rank 2 must possess two functional arbitrarinesses then we can

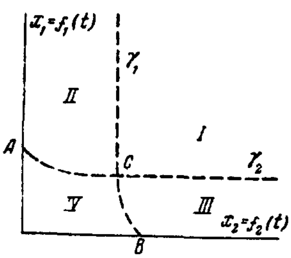


Fig. 1.

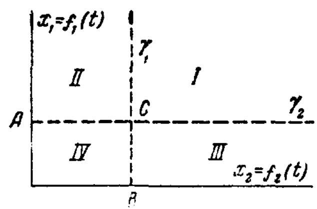


Fig. 2.

satisfy the boundary condition $u_i = f_i(t)$, possessing that same arbitrariness.

The condition which fixes the function $\phi(u_1, u_2)$ is the condition of uninterrupted connection (or continuity) of the solution in region IV to the solutions in regions II and III.

It is easy to see that they have the form

$$\varphi(u_1, 0) = \frac{1}{\gamma-1}\left[\frac{\gamma-1}{2}u_1 + c_0\right]^2, \qquad \varphi(0, u_2) = \frac{1}{\gamma-1}\left[\frac{\gamma-1}{2}u_2 + c_0\right]^2 \tag{4.5}$$

In this manner, $\phi$ must be identified with that solution of equation $L(\phi) = 0$ which satisfies the boundary conditions (4.5). We have the problem of Goursat for quasi-linear equations of the second order.

5. Let us consider isothermal gases. Then

$$\gamma = 1,\ \theta = \ln \rho,\ a^2 = RT = \text{const}.$$

For simplicity, we assume $a^2 = 1$. All the results of preceding problems are automatically transferred to the isothermal gases.

The boundary problem of the preceding section has the following form:

$$L(\varphi) = (1-\varphi_2^2)(\varphi_{11}+1) + 2\varphi_1\varphi_2\varphi_{12} + (1-\varphi_1^2)(\varphi_{22}+1) = 0 \tag{5.1}$$

$$\varphi(u_1, 0) = u_1 + \theta_0, \qquad \varphi(0, u_2) = u_2 + \theta_0 \tag{5.2}$$

It is easy to see that the solution of this problem is the function

$$\varphi = u_1 + u_2 + \theta_0 \tag{5.3}$$

Function $X(u_1, u_2)$ must satisfy equation

$$(1-\varphi_2^2)\frac{\partial^2 X}{\partial u_1^2} + 2\varphi_1\varphi_2\frac{\partial^2 X}{\partial u_1 \partial u_2} + (1-\varphi_1^2)\frac{\partial^2 X}{\partial u_2^2} = 0 \tag{5.4}$$

which in the case of (5.3) takes the form:

$$\frac{\partial^2 X}{\partial u_1 \partial u_2} = 0 \tag{5.5}$$

From this follows at once $x_i = x_i(u_i, t)$. The picture of the motion assumes the form shown in Fig.2.

Sections AC and BC represent continuations of lines $\gamma_2$ and $\gamma_1$, respectively. In this manner, regions I to IV are bounded by mutually-orthogonal straight lines.

The equations of the straight lines $\gamma_1$ and $\gamma_2$ respectively are

$$x_1 = t, \qquad x_2 = t \tag{5.6}$$

Further we have in region I

$$u_1 = u_2 = 0, \tag{5.7}$$

in region II $\quad u_2 = 0, \quad u_1 = g_1(x_1, t)$ (5.8)

in region III $\quad u_1 = 0, \quad u_2 = g_2(x_2, t)$ (5.9)

in region IV $\quad u_1 = g_1(x_1, t), \quad u_2 = g_2(x_2, t)$ (5.10)

in region I to IV $\quad \theta = u_1 + u_2 + \theta_0$ (5.11)

Function $g_i(x_i, t)$ is the fundamental solution of equation

$$u_i = F_i\,[x_i - (u_i + 1)\,t] \tag{5.12}$$

where function $F_i(t)$ is related to $f_i(t)$ by the equation

$$f_i'(t) = F_i\,[f_i(t) - (f_i' + 1)\,t] \tag{5.13}$$

Formulae (5.6) to (5.13) give a complete solution of the problem under consideration of the motion of a isothermal gas enclosed inside the straight corner in the assumed absence of strong discontinuities. Straight lines $\gamma_1$, $\gamma_2$ appear to be lines of weak discontinuities (discontinuities of derivatives $\partial u_i/\partial x_j$ $\partial u_i/\partial t$, $\partial\theta/\partial x_j$, $\partial\theta/\partial t$).

Of course, there may still be other discontinuities besides in the regions II, III and IV.

For instance when

$$f_i(t) = c_i < 0 \tag{5.14}$$

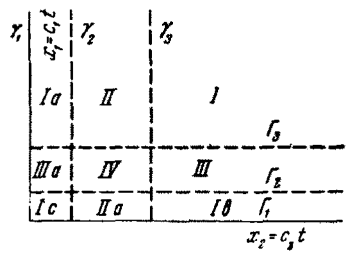


**Fig. 3.**

The picture of the motion assumes the form shown in Fig. 3.

In the regions I, Ia, Ib, Ic we have a motion with constant parameters:

$$\begin{array}{lllllll} (I) & u_1 = 0 & u_2 = 0; & \qquad (Ib) & u_1 = 0, & u_2 = c_2 & \\ (Ia) & u_1 = c_1, & u_2 = 0; & \qquad (Ic) & u_1 = c_1, & u_2 = c_2 & (5.15) \end{array}$$

In the regions II, IIa, III, IIIa we have plane travelling waves (Riemann waves):

$$\begin{array}{lll} (II) & u_1 = x_1/t - 1, & u_2 = 0 \\ (IIa) & u_1 = x_1/t - 1, & u_2 = c_2 \\ (III) & u_1 = 0, & u_2 = x_2/t - 1 \\ (IIIa) & u_1 = c_1, & u_2 = x_2/t - 1 \end{array} \tag{5.16}$$

In region IV we have a travelling (conical) wave of rank 2:

$$u_1 = x_1/t - 1, \qquad u_2 = x_2/t - 1 \tag{5.17}$$

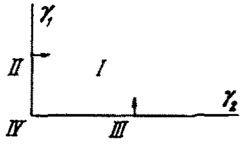


**Fig. 4.**

Conditions in all regions are governed by the relation (5.11). The lines $\gamma_i$, $\Gamma_i$ dividing the mentioned regions move according to the law

$$\begin{array}{llll} (\gamma_1) & x_1 = c_1 t, & (\Gamma_1) & x_2 = c_2 t \\ (\gamma_2) & x_1 = (c_1 + 1)t, & (\Gamma_2) & x_2 = (c_2 + 1) \\ (\gamma_3) & x_1 = t, & (\Gamma_3) & x_2 = t \end{array} \tag{5.18}$$

Generally the motion is conical (self similar or centered wave).

**6.** Let us now consider motions in which strong discontinuities may occur, confining ourselves to the case of an isothermal gas.

The Hugoniot conditions for an isothermal gas with $a^2 = 1$ have the form:

$$u_1 - u_0 = M_0 - \frac{1}{M_0}, \qquad \theta_1 - \theta_0 = \ln M_0^2 \qquad M_0 = D - u_0 \tag{6.1}$$

Here the index (0) corresponds to conditions ahead of the front, the index (1) refers to conditions behind the shock waves. It is easy to see that the configuration of two steady and compatible shock fronts move with constant speed through a gas at rest in mutually orthogonal directions (Fig.4).

If the speed of front $\gamma_1$ equals $D_1$ and the speed of front $\gamma_2$ equals $D_2$ then we have by virtue of (6.1)

$$\begin{array}{llll} u_1 = D_1 - \dfrac{1}{D_1}, & u_2 = 0, & \theta = \theta_0 + \ln D_1^2 & \text{in region II} \\ u_1 = 0, & u_2 = D_2 - \dfrac{1}{D_2}, & \theta = \theta_0 + \ln D_2^2 & \text{in region III} \\ u_1 = D_1 - \dfrac{1}{D_1}, & u_2 = -\dfrac{1}{D_2} + D_2, & \theta = \theta_0 + \ln D_1^2 + \ln D_2^2 & \text{in region IV} \end{array} \tag{6.2}$$

The above relations take into account the compatibility conditions in all shock fronts.

Steady and compatible also is the configuration of shock wave $\gamma_1$, and the Riemann travelling waves $(\gamma_2, \gamma_3)$ in the case where they travel in mutually orthogonal directions. (Fig.5).

The motion in that case is characterized in the following manner:

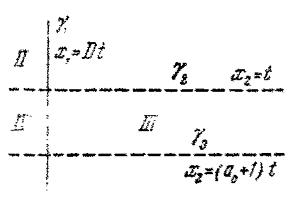


Fig. 5.

$$\begin{array}{llll} (I) & u_1 = 0, & u_2 = 0, & \theta = \theta_0 \\ (II) & u_1 = D - \dfrac{1}{D}, & u_2 = 0, & \theta = \theta_0 + \ln D^2 \\ (III) & u_1 = 0, & u_2 = \dfrac{x_2}{t} - 1, & \theta = \theta_0 + u_2 \\ (IV) & u_1 = D - \dfrac{1}{D}, & u_2 = \dfrac{x_2}{t} - 1, & \theta = u_2 + \ln D^2 + \theta_0 \end{array} \tag{6.3}$$

The conditions of compatibility are fulfilled on all boundaries. The motion considered in Figs. 4 and 5 can be obtained when one of the edges of the straight corner moves with a constant positive speed and the other either also with a constant positive speed or according to a certain law $x = f(t)$ insuring the absence of strong discontinuities. Summarizing investigations of sections 5 and 6, one can formulate the following theorem:

*Theorem 3.* Let the edges of a straight corner move according to a law

$x_i = f_i(t)$, $i = 1,2$. Then the $x_1$, $x_2$ plane is cut by two mutually-orthogonal straight lines $\gamma_1$, $\gamma_2$ into four regions I to IV (Fig.2), so that the following regime of motion prevails:

$$
\begin{array}{llll}
u_1 = 0, & u_2 = 0 & \theta = \theta_0 & \text{in region I} \\
u_1 = g_1(x_1, t), & u_2 = 0 & \theta = \theta_0 + u_1 + c_1 & \text{in region II} \\
u_1 = 0, & u_2 = g_2(x_2 t) & \theta = \theta_0 + u_2 + c_2 & \text{in region III} \\
u_1 = g_1(x_1, t), & u_2 = g_2(x_2, t) & \theta = \theta_0 + u_1 + u_2 + c_3 & \text{in region IV}
\end{array}
$$

This representation is true even for the case when for one $i f_i(t) = c_i > 0$. Then the respective boundary $\gamma_i$ is a shock wave, proceeding with constant speed: in the remaining cases $\gamma_i$ is a line of weak discontinuity. When shock waves are absent, $c_1 = c_2 = c_3 = 0$.

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